MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 6 - *SOLUTIONS*

Let f be real-valued function whose domain is a subset of the real numbers. We say that f is L-Lipschitz if for every pair of points x, y in the domain of f, $|f(x) - f(y)| \leq L\,|x - y|$.

Problem 1 (80 points)**.** Prove or find a counterexample for each:
(a) If f is uniformly continuous, then f is L-Lipschitz for some $L > 0$
(b) If f is L-Lipschitz for some $L > 0$, then f is uniformly continuous
(c) If f and g are L-Lipschitz, then there exists an L' such that $f + g$ is L'-Lipschitz
(d) If f is L-Lipschitz, then there exists some L' such that $g(x) := f(x)^2$ is L'-Lipschitz

Solution.
(a) This is **false**. The function $f(x) = \sqrt{x}$ on $[0, 1]$ must be uniformly continuous, since it is continuous on a compact interval. However it is not L-Lipschitz for any L. Indeed, fix $L > 0$. Then if $x = 0$ and $y = 1/(2L)^2$,

$$|f(x) - f(y)| = \left|0 - \sqrt{1/(2L)^2}\right| = 1/(2L) = 2L\left|0 - 1/(2L)^2\right| > L\,|x - y|\,.$$

Hence, $\sqrt{x}$ is not L-Lipschitz.
(b) This is **true**. Let f be L-Lipschitz, and fix $\varepsilon > 0$. Set $\delta = \varepsilon/L$. Then if $|x - y| < \delta$,

$$|f(x) - f(y)| \leq L\,|x - y| < L \cdot \varepsilon/L = \varepsilon.$$

Hence f is uniformly continuous.
(c) This is **true**. We claim that if f and g are L-Lipschitz then $f + g$ is $2L$-Lipschitz. Fix x and y in the domain of f and g. Then

$$\begin{aligned}|(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \leq L\,|x - y| + L\,|x - y| = 2L\,|x - y|\,.\end{aligned}$$

(d) This is **false**. Consider $f(x) = x$ defined on $\mathbb{R}$. Then f is 1-Lipschitz since $|f(x) - f(y)| = 1 \cdot |x - y|$. But $g(x) = x^2$ is not L-Lipschitz for any L, since if $x = L$ and $y = L + 1$, then

$$|f(x) - f(y)| = \left|(L+1)^2 - L^2\right| = 2L + 1 > L \cdot 1 = L\,|x - y|\,.$$

□

Problem 2 (20 points)**.** Show that if $f : [a, b] \to [c, d]$ is continuous and has an inverse, then either f is increasing or f is decreasing.

Solution. Assume that f is continuous and has an inverse. Then there exists a unique $x \in [a, b]$ such that $f(x) = c$. We claim that either $x = a$ or $x = b$. Indeed, assume for a contradiction that $a < x < b$. Then $f(a)$ and $f(b)$ are both greater than c, since only one element of $[a, b]$ can map to c. Choose some z such that $c < z < \min\{f(a), f(b)\}$. By the intermediate value theorem, there exists $y_1 \in (a, x)$ such that $f(y_1) = z$ and $y_2 \in (x, b)$ such that $f(y_2) = z$. Thus, f is not invertible, so either $f(a) = c$ or $f(b) = c$.

In the case that $f(a) = c$, we claim that f is increasing. We again proceed by contradiciton. Assume there exists $a \leq x < y$ such that $f(x) > f(y)$. Choose some z such that $f(y) < z < f(x)$. Then since $f(a) = c \leq f(x)$, by the intermediate value theorem applied ot f on $[a, x]$, there exists

$y_1 \in (a, x)$ such that $f(y_1) = z$. Similarly, there exists $y_2 \in (x, y)$ such that $f(y_2) = z$. This is a contradiction to the invertibility of f.

A parallel argument works when $f(a) = d$, in which case we conclude that f is decreasing. $\square$